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Edge-removing games of star type

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Abstract

We define a new game, which is called an edge-removing game of star type. This game is played by two players on a graph G. Each player in turn removes a set of edges which induces a star, and the player who removes the last edge wins. This game is a generalization of the game of classical Nim and classical Kayles since if we play this game on a graph consisting of stars or one consisting of paths, then this game is equivalent to the game of classical Nim or classical Kayles. We give some results on this game.

1. Introduction

We first define a new game played on graphs. Let G be a finite graph without loops or multiple edges, and $\mathscr{H}$ a set of graphs. This game is played by two players on the graph G. Each player in turn removes a set of edges which induces a graph isomorphic to a graph in $\mathscr{H}$. The winner is a player who removes edges such that the resulting graph contains no graph of $\mathscr{H}$, that is, the player who cannot move loses. We call this game an *edge-removing game of normal $\mathscr{H}$ type*. If we change the rule to one where the player who removes the last edges loses, then the game is called an *edge-removing game of misere $\mathscr{H}$ type*. In this paper, we shall discuss only games of normal type.

We call the complete bipartite graph $K_{1,n} = K(1,n)$ the *star* of order $n+1$, and denote by P_n the path of order n. If $\mathscr{H}$ is the set of all stars, then we call this game the *edge-removing game of normal star type*, or simply *ER-game of star type*. If we play ER-game of star type on a graph consisting of some stars, then this game is nothing but the game of classical Nim [1]. Similarly, ER-game of star type played on a graph consisting of some paths is equivalent to the game called classical Kayles [1]. So ER-game of star type is a generalization of these two games. In this paper we give some results on ER-game of star type played on double stars and forks.

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2. ER-game of star type played on doubles stars

In order to solve ER-game of normal $\mathscr{H}$ type played on a graph G, it suffices to determine the Sprague–Grundy number $g(G)$ of G, which is often called the Grundy number [1–3]. The Grundy number is defined inductively as follows: If a graph G_1 contains no graph of $\mathscr{H}$, then $g(G_1) = 0$. Let $H_1, H_2, \ldots, H_m$ be the set of all graphs which can be obtained from a graph G by one move. Then

$$g(G) = \min\{\{0,1,2,3,\ldots\} - \{g(H_i) \mid 1 \leqslant i \leqslant m\}\}.$$

By this definition, we can easily show that $g(G) \leqslant |E(G)|$ by induction on $|E(G)|$. It is well-known that if a graph G consists of the components $D_1, \ldots, D_r$, then

$$\begin{aligned} g(G) &= \text{the nim-sum of } g(D_1), g(D_2), \ldots, g(D_r). \\ &= g(D_1) \dotplus g(D_2) \dotplus \cdots \dotplus g(D_r). \end{aligned}$$

Namely, if

$$g(D_k) = \sum_{i \geqslant 0} x_k(i) 2^i, \quad x_k(i) \in \{0, 1\},$$

then

$$g(G) = \sum_{i \geqslant 0} y(i) 2^i, \quad y(i) \equiv \sum_{k=1}^{r} x_k(i) \pmod 2 \quad \text{and} \quad y(i) \in \{0, 1\}.$$

Moreover, it is easy to see that the player going second can win if and only if $g(G) = 0$.

We denote by $(\ldots)$ an order set, that is, $(x_1, x_2, \ldots, x_k) = (y_1, y_2, \ldots, y_k)$ means that $x_i = y_i$ for all i, $1 \leqslant i \leqslant k$.

Theorem A (Ball and Coxeter [1], Berlekamp et al. [2] and Hitotumatu [3]). *The Grundy numbers of stars $K_{1,n}$ and paths P_n of ER-game of star type are given by the following statements.*

(i) $g(K_{1,n}) = n$.

(ii) $g(P_{n+12}) = g(P_n)$ *if* $n \geqslant 72$, *and* $(g(P_k) \mid 72 \leqslant k < 84) = (7,4,1,2,8,1,4,7,2,1,8,2)$.

For convenience, we denote by $K_{1,0}$ a graph with one vertex and no edge. The double star $\mathrm{DS}(n, m)$ is a graph obtained from two stars $K_{1,n}$ and $K_{1,m}$ by joining their two centers by a new edge. Then the order of $\mathrm{DS}(n, m)$ is $n + m + 2$ and its size is $n + m + 1$. We now give a conjecture on the grundy numbers of double stars.

Conjecture B. Suppose that ER-game of star type is played on a double star $\mathrm{DS}(n, m)$. Then

(i) For every positive odd integer n, there exists an integer $M = M(n)$ for which $g(\mathrm{DS}(n, m)) = n + m + 1$ if $m \geqslant M$.

(ii) For every positive even integer n, there exists integers $p = p(n)$ and $M = M(n)$ for which $g(\mathrm{DS}(n, m + p)) = g(\mathrm{DS}(n, m)) + p$ if $m \geqslant M$.

We shall show that the conjecture is true if $n = 2^k - 1$, $n = 2^k$ or $1 \leqslant n \leqslant 10$. Moreover, by making use of computer, we observe that if $n < 50$ and $m \leqslant 5000$ then the conjecture holds and that $M(n) < 800$ except $n = 33\,(M(n) = 1953)$, $n = 34\,(M(n) = 2141)$ and $n = 48\,(M(n) = 2157)$; furthermore, we may give a conjecture on $p(n)$ that $p(n) = 2^{k+1}$ if $2^k \leqslant n < 2^{k+1}$ except $n = 24$ $(p(n) = 64)$.

Theorem 1. *Suppose that ER-game is played on a double star. Then*

(i) *For every integers $k \geqslant 1$ and $m \geqslant 0$, we have*

$$g(\mathrm{DS}(2^k - 1, m)) = 2^k + m.$$

(ii) *For every integers $k \geqslant 1$ and $h \geqslant 0$, we have*

$$g(\mathrm{DS}(2^k, h2^{k+1} + s)) = h2^{k+1} + 2^k + s + 1,$$

where $-1 \leqslant s \leqslant 2^k - 1$, *and*

$$g(\mathrm{DS}(2^k, h2^{k+1} + 2^k + s)) = h2^{k+1} + s + 1,$$

where $0 \leqslant s \leqslant 2^k - 2$. *In particular,*

$$g(\mathrm{DS}(2^k, m + 2^{k+1})) = g(\mathrm{DS}(2^k, m)) + 2^{k+1} \quad \textit{for all } m \geqslant 0.$$

Proof. For convenience, we denote the star $K_{1,l}$ by $K(1, l)$. We first prove statement (i) by induction on m. Since a double star $\mathrm{DS}(2^k - 1, 0)$ is a star $K(1, 2^k)$, $g(\mathrm{DS}(2^k - 1, 0)) = 2^k$ by Theorem A.

Suppose that $1 \leqslant m < 2^k$. For every integer j, $0 \leqslant j < 2^k$, let $r = m \dotplus j$. Then $0 \leqslant r < 2^k$ and $r \dotplus m = j$. We can remove a star from the double star $\mathrm{DS}(2^k - 1, m)$ such that the resulting graph is $K(1, r) \cup K(1, m)$, whose Grundy number is $r \dotplus m = j$. By the induction hypothesis, we have that $g(\mathrm{DS}(2^k - 1, y)) = 2^k + y$ for every $0 \leqslant y < m$. Therefore $g(\mathrm{DS}(2^k - 1, m)) \geqslant 2^k + m$. Since $g(\mathrm{DS}(2^k - 1, m)) \leqslant |E(\mathrm{DS}(2^k - 1, m))| = 2^k + m$, we can conclude that $g(\mathrm{DS}(2^k - 1, m)) = 2^k + m$.

Next assume that $2^k \leqslant m$. For every integer j, $0 \leqslant j < 2^k$, let $r = (2^k - 1) \dotplus j$. Then $0 \leqslant r < 2^k$ and $(2^k - 1) \dotplus r = j$. We can remove a star from the double star $\mathrm{DS}(2^k - 1, m)$ such that the resulting graph is $K(1, 2^k - 1) \cup K(1, r)$, whose Grundy number is $(2^k - 1) \dotplus r = j$. By the same argument as above, we can also show that $g(\mathrm{DS}(2^k - 1, m)) = 2^k + m$.

In order to prove statement (ii) we need to show that the following equation holds:

$$g(\mathrm{DS}(h2^{k+1} - 1, m)) = h2^{k+1} + m$$

for every integers $0 \leqslant m \leqslant 2^k$ and $h \geqslant 1$. We prove the above equation by induction on m. If $m = 0$ then the equation holds by Theorem A, and so we may assume $m \geqslant 1$. Let $0 \leqslant j < 2^k$ and $0 \leqslant t \leqslant h - 1$. If $m < 2^k$, then $r = j \dotplus m < 2^k$, and we can remove stars from $\mathrm{DS}(h2^{k+1} - 1, m)$ such that the resulting graphs are $K(1, t2^{k+1} + r) \cup K(1, m)$ and $K(1, t2^{k+1} + 2^k + r) \cup K(1, m)$, whose Grundy numbers are $t2^{k+1} + j$ and $t2^{k+1} + 2^k + j$, respectively. If $m = 2^k$ then we can remove stars from $\mathrm{DS}(h2^{k+1}-1, m)$ such that the resulting graphs are $K(1, t2^{k+1}+j) \cup K(1, 2^k)$ and $K(1, t2^{k+1}+2^k+j) \cup K(1, 2^k)$, whose Grundy numbers are $t2^{k+1}+2^k+j$ and $t2^{k+1}+j$, respectively. By the induction hypothesis, we have that $g(\mathrm{DS}(h2^{k+1}-1, y)) = h2^{k+1}+y$ for every $0 \leqslant y < m$. Thus $g(\mathrm{DS}(h2^{k+1}-1, m)) \geqslant h2^{k+1}+m$, and so $g(\mathrm{DS}(h2^{k+1}-1, m)) = 2^{k+1}+m$.

We now prove statement (ii) by induction on $h2^{k+1}+s$ or $h2^{k+1}+2^k+s$. By Theorem A and the above equation, $g(\mathrm{DS}(2^k, 0)) = 2^k+1$ and $g(\mathrm{DS}(2^k, h2^{k+1}-1)) = h2^{k+1}+2^k$. Consider a double star $\mathrm{DS}(2^k, h2^{k+1}+s)$, $0 \leqslant h, 0 \leqslant s \leqslant 2^k-1$. For every integers $0 \leqslant t \leqslant h-1$, and $0 \leqslant x < 2^k$, we have that

$$g(K(1, 2^k)) \dotplus g(K(1, t2^{k+1}+2^k+x)) = t2^{k+1}+x$$

and

$$g(K(1, 2^k)) \dotplus g(K(1, t2^{k+1}+x)) = t2^{k+1}+2^k+x.$$

Moreover,

$$g(K(1, x' \dotplus s)) \dotplus g(K(1, h2^{k+1}+s)) = h2^{k+1}+x' \quad \text{for } 0 \leqslant x' \leqslant 2^k,$$

and for every integer y, $0 \leqslant y < s$, it follows from the induction hypothesis that

$$g(\mathrm{DS}(2^k, h2^{k+1}+y)) = h2^{k+1}+2^k+y+1.$$

Therefore $g(\mathrm{DS}(2^k, h2^{k+1}+s)) \geqslant h2^{k+1}+2^k+s+1$, and thus $g(D(2^k, h2^{k+1}+s)) = h2^{k+1}+2^k+s+1$.

We next consider a double star $\mathrm{DS}(2^k, h2^{k+1}+2^k+s)$, $0 \leqslant h, 0 \leqslant s \leqslant 2^k-2$. By the same argument as above, we can easily show that

$$\{g(K(1, 2^k)) \dotplus g(K(1, y)) \mid 0 \leqslant y \leqslant h2^{k+1}+2^k+s\}$$
$$= \{0, 1, 2, \ldots, h2^{k+1}+s\} \cup \{h2^{k+1}+2^k, \ldots, (h+1)2^{k+1}-1\}.$$

Thus $g(\mathrm{DS}(2^k, h2^{k+1}+2+s)) \geqslant h2^{k+1}+s+1$. Let t be an integer such that $0 < t < 2^k$. Then it is obvious that $g(K(1, t) \dotplus g(K(1, h2^{k+1}+2^k+s)) \geqslant h2^{k+1}+2^k$. Moreover, we have that $\mathrm{DS}(t, h2^{2k+1}+2^k+s)$ contains $K(1, t) \cup K(1, h2^{k+1}+r)$, $r = t \dotplus (s+1)$, whose Grundy number is $h2^{k+1}+s+1$. Therefore $g(\mathrm{DS}(t, h2^{k+1}+2^k+s)) \neq h2^{k+1}+s+1$. Consequently, we can conclude that $g(\mathrm{DS}(2^k, h2^{k+1}+2^k+s)) = h2^{k+1}+s+1$. □

Theorem 2. *The Grundy numbers of double stars* $\mathrm{DS}(n, m)$, $n \leqslant 10$, *of ER-game of star type are given by the following statements.*

(i) *If* $n = 0$, $n = 1$, $n = 3$, $n = 5$ *and* $m \geqslant 15$, $n = 7$ *or* $n = 9$ *and* $m \geqslant 95$ *then*

$$g(\mathrm{DS}(n, m)) = n+m+1.$$

(ii) *Let* $p = 4, 8$ *or* 16 *according as* $n = 2$, $n = 4, 6$ *or* $n = 8, 10$. *Suppose that* $m \geqslant 15$ *if* $n = 6$, *and* $m \geqslant 110$ *if* $n = 10$. *Then*

$$g(\mathrm{DS}(n, m+p)) = g(\mathrm{DS}(n, m)) + p.$$

Note that Theorem 2 holds for $n = 0, 1, 2, 3, 4, 7, 8$ by Theorem A and Theorem 1. We shall prove the following proposition instead of remaining statement (ii) of the Theorem 2.

Proposition 3. *Consider* $g(\mathrm{DS}(n, m))$ *with* $n = 2, 4, 6, 8$ *or* 10. *Let* t *and* s *be integers such that* $0 \leqslant t$, *and* $0 \leqslant s < 4$, $0 \leqslant s < 8$, *or* $0 \leqslant s < 16$ *according as* $n = 2$, $n = 4$, 6 *or* $n = 8$, 10. *Then the following statements hold.*

(i) $g(\mathrm{DS}(2, 4t + s)) = 4t + 3,\ 4t + 4,\ 4t + 1,\ 4t + 6$ *if* $s = 0, 1, 2, 3$, *respectively.*

(ii) $g(\mathrm{DS}(4, 8t + s)) = 8t + 5, 8t + 6, 8t + 7, 8t + 8, 8t + 1, 8t + 2, 8t + 3, 8t + 12$ if $s = 0, 1, 2, \ldots, 7$ *respectively.*

(iii) $g(\mathrm{DS}(6,\ 15 + 8t + s)) = 8t + 22,\ 8t + 23,\ 8t + 24,\ 8t + 21,\ 8t + 26,\ 8t + 25,\ 8t + 28,\ 8t + 27$ *if* $s = 0, 1, 2, \ldots, 7$ *respectively.*

(iv) $g(\mathrm{DS}(8, 16t + s)) = 8t + s + 9, 8t + s - 7$ *or* $8t + 24$ *if* $0 \leqslant s \leqslant 7, 8 \leqslant s \leqslant 14$ *or* $s = 15$, *respectively.*

(v) $g(\mathrm{DS}(10, 110 + 8t + s)) = 8t + 117,\ 8t + 122,\ 8t + 123,\ 8t + 124,\ 8t + 121,\ 8t + 126,\ 8t + 127,\ 8t + 128,\ 8t + 125,\ 8t + 130,\ 8t + 119,\ 8t + 132,\ 8t + 129$ *if* $s = 0, 1, \ldots, 15$, *respectively.*

Proof. We simultaneouly prove (i) of Theorem 2 and Proposition 3 by induction on n, and each statement on fixed n is proved by induction on m. Moreover, we shall prove only the statements on $n = 5$ of Theorem 2 and (iii) of Proposition 3 since other cases can be proved similarly.

Since $\mathrm{DS}(0, m) = K_{1,m+1}$, we have $g(\mathrm{DS}(0, m)) = m + 1$ by Theorem A. We consider two cases.

Case 1: $n = 5$. We prove that

$$g(\mathrm{DS}(5, m)) = m + 6 \quad \text{for all } m \geqslant 15.$$

By the aid of computer, we have $(g(\mathrm{DS}(5, j)) \mid 0 \leqslant j \leqslant 14) = (6, 7, 8, 9, 2, 3, 11, 13, 14, 15, 16, 17, 10, 18, 19)$. Let $m \geqslant 15$. It follows from the definition of Grundy number that

$$\begin{aligned} g(\mathrm{DS}(5, m)) = \mathrm{Min}\{&\{0, 1, 2, \ldots, \} \\ &- \{g(\mathrm{DS}(i, m)) \mid 0 \leqslant i < 5\} - \{g(K_{1,i}) \dot{+} g(K_{1,m}) \mid 0 \leqslant i \leqslant 5\} \\ &- \{g(\mathrm{DS}(5, j)) \mid 0 \leqslant j \leqslant m\} - \{g(K_{1,5}) \dot{+} g(K_{1,j}) \mid 0 \leqslant j \leqslant m\}\}. \end{aligned}$$

Since $\{0, 1, 2, \ldots, m + 5\} - \{g(\mathrm{DS}(5, j)) \mid 0 \leqslant j < m\} = \{0, 1, 4, 5, 12, 20\}$, $\{g(K_{1,5}) \dot{+} g(K_{1,j}) \mid j = 5, 4, 1, 0, 9\} = \{5 \dot{+} j \mid j = 5, 4, 1, 0, 9\} = (0, 1, 4, 5, 12\}$ and $g(\mathrm{DS}(4, 15)) = 20$, we obtain that $g(\mathrm{DS}(5, m)) \geqslant 6 + m$.

By the induction hypothesis on $n \leqslant 4$, we can show that $g(\mathrm{DS}(i, m)) \leqslant 5 + m$ if $i \leqslant 4$. It is obvious that $x \dot{+} y \leqslant x + y$, and so $g(K_{1,i}) \dot{+} g(K_{1,j}) = i \dot{+} j \leqslant i + j \leqslant 5 + m$. Therefore $g(\mathrm{DS}(5, m)) \leqslant 6 + m$, and we can conclude that $g(\mathrm{DS}(5, m)) = 6 + m$.

Case 2: $n = 6$. We shall prove statement (iii) of Proposition 1 by induction on m. By the aid of computer, we have $(g(\mathrm{DS}(6,j)) \mid 0 \leqslant j \leqslant 14) = (7, 8, 5, 10, 3, 11, 1, 14, 15, 16, 13, 18, 17, 19, 9)$ and the statement holds for $m \leqslant 22$. Let $m = 15 + 8t + s$, $t \geqslant 1$. It follows from the definition of Grundy number that

$$\begin{aligned} g(\mathrm{DS}(6,m)) = \mathrm{Min}\{&\{0,1,2,\dots\} \\ &- g(\mathrm{DS}(i,m)) \mid 0 \leqslant i < 6\} - \{g(K_{1,i}) \dotplus g(K_{1,m}) \mid 0 \leqslant i \leqslant 6\} \\ &- \{g(\mathrm{DS}(6,j)) \mid 0 \leqslant j < m\} - \{g(K_{1,6}) \dotplus g(K_{1,j}) \mid 0 \leqslant j \leqslant m\}\}. \end{aligned}$$

Since $\{g(\mathrm{DS}(6,j)) \mid 15 \leqslant j < 15 + 8t\}$ is a set of consecutive integers $\{21, 22, \dots, 8(t-1)+28\}$, we have $\{0,1,2,\dots,8(t-1)+28\} - \{g(\mathrm{DS}(6,j)) \mid 0 \leqslant j < 15 + 8t\} = \{0,1,2,\dots,20\} - \{g(\mathrm{DS}(6,j)) \mid 0 \leqslant j < 15\} = \{0, 2, 4, 6, 12, 20\}$. It is obvious that $\{g(K_{1,6}) \dotplus g(K_{1,j}) \mid j = 6,4,2,0,10,18\} = \{0,2,4,6,12,20\}$, and thus we have $g(\mathrm{DS}(6, 15 + 8t + s)) \geqslant 8(t-1) + 28 + 1 = 8t + 21$.

Since $g(\mathrm{DS}(5, 15 + 8t)) = 8t + 21$, we have $g(\mathrm{DS}(6, 15 + 8t)) \geqslant 8t + 22$. Similarly, $g(\mathrm{DS}(5, 15 + 8t + 1)) = 8t + 22$ and $g(\mathrm{DS}(4, 15 + 8t + 1)) = 8t + 21$ imply $g(\mathrm{DG}(6, 15 + 8t + 1)) \geqslant 8t + 23$. Grundy numbers $g(\mathrm{DS}(5, 15 + 8t + 2)) = 8t + 23$, $g(\mathrm{DS}(4, 15 + 8t + 2)) = 8t + 22$ and $g(\mathrm{DS}(3, 15 + 8t + 2)) = 8t + 21$ imply $g(\mathrm{DS}(6, 15 + 8t + 2)) \geqslant 8t + 24$. On the other hand, if $0 \leqslant s \leqslant 2$ then we can show that $g(\mathrm{DS}(6, 15 + 8 + s)) \leqslant 8t + 22 + s$ by $g(K_{1,i}) \dotplus g(K_{1,j}) \leqslant i + j \leqslant 8t + 21 + s$. Thus $g(\mathrm{DS}(6, 15 + 8t + s)) = 8t + 22 + s$ if $0 \leqslant s \leqslant 2$. If $s = 3$ then $m = 15 + 8t + 3$ and $8t + 21$ is not contained in $\{g(\mathrm{DS}(i,m)) \mid 0 \leqslant i < 6\} \cup \{g(K_{1,i}) \dotplus g(K_{1,m}) \mid 0 \leqslant i \leqslant 6\} \cup \{g(K_{1,6}) \dotplus g(K_{1,j} \mid 0 \leqslant j \leqslant m\}$. Hence $g(\mathrm{DS}(6, 15 + 8t + 3)) = 18t + 21$. By the same argument as above, we can show that $g(\mathrm{DS}(6, 15 + 8t + s) = 8t + 26, 8t + 25, 8t + 28, 8t + 27$ if $s = 4, 5, 6, 7$, respectively. $\square$

A *fork* $F(n, m)$ is defined to be a graph which is obtained from a star $K_{1,n}$ and a path P_m by joining the center of the star to one of the end vertices of the path by a new edge. Then the order of $F(n, m)$ is $n + m + 1$ and its size is $n + m$. Note that $F(0, m) = P_{m+1}$, $F(1, m) = P_{m+2}$, $F(n, 0) = K_{1,n}$ and $F(n, 1) = K_{1,n+1}$ and these Grundy numbers are given by Theorem A. We give a conjecture on the Grundy numbers $g(F(n, m))$ of forks $F(n, m)$. This conjecture is true if $n \leqslant 10$ or $m \leqslant 10$.

Conjecture C. (i) For every positive integer n, there exists an integer $M = M(n)$ for which $g(F(n, m + 12)) = g(F(n, m))$ if $n \geqslant M$.

(ii) For every positive even integer m, there exists integers $p = p(m)$ and $M = M(m)$ for which $g(F(n + p, m)) = g(F(n, m)) + p$ if $n \geqslant M$.

We conclude this paper with the following problem.

Problem. Is it possible to solve ER-games of the following $\mathscr{H}$ type on certain class of graphs: $\mathscr{H}$ is the set of all cycles, $\mathscr{H}$ is the set of all trees, $\mathscr{H}$ is the set of all matchings, $\mathscr{H}$ is the set of all forests, and so on?

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